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Fractal-to-Euclidean crossover of the isotropy restoration feature in a family of fractal resistor networks

Jianjun Xu, Zhifang Lin, and Zuowei Wang
Department of Physics, Fudan University, Shanghai 200433, China
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Fractal with microscopic anisotropy shows a unique type of macroscopic isotropy restoration phenomenon that is absent in Euclidean space [M. T. Barlow *et al.*, Phys. Rev. Lett. **75**, 3042]. In this paper the isotropy restoration feature is considered for a family of two-dimensional Sierpiński gasket type fractal resistor networks. A parameter ξ is introduced to describe this phenomenon. Our numerical results show that ξ satisfies the scaling law $\xi \sim l^{-\alpha}$, where l is the system size and α is an exponent independent of the degree of microscopic anisotropy, characterizing the isotropy restoration feature of the fractal systems. By changing the underlying fractal structure towards the Euclidean triangular lattice through increasing the side length b of the gasket generators, the fractal-to-Euclidean crossover behavior of the isotropy restoration feature is discussed. [S1063-651X(98)09306-4]

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Recently, Barlow and co-workers reported a new type of restoration of macroscopic isotropy in fractal systems with microscopic anisotropy [1]. The phenomenon is unique in the sense that it is absent in uniform media such as regular lattices or Euclidean spaces, while it is universal since it can be observed in many physical setups on a wide class of fractal systems. In the present study we introduce a parameter ξ to describe this isotropy restoration phenomenon by studying a family of two-dimensional Sierpiński gasket (SG) type fractal resistor networks, which are characterized by generators of side length b. Our numerical results show that the anisotropy parameter ξ obeys the scaling relation $\xi \sim l^{-\alpha}$, where l is the system size whereas α is an exponent independent of the degree of microscopic anisotropy and thus is believed to be able to characterize the isotropy restoration feature of the fractal systems.

On the other hand, the isotropy restoration feature is absent in ordinary Euclidean space. Thus one may rightfully address the question of what happens at the fractal-to-Euclidean crossover [2]. A similar question has been posed and investigated for the thermodynamic properties of the Ising model [3]. To study the fractal-to-Euclidean crossover behavior of the isotropy restoration feature, we calculate the isotropy restoration exponent α for a family of SG type resistor networks with increasing generator side length b. It is found that α is monotonously decreasing with the fractal dimension d_f and spectral dimension d_s of the underlying fractal structure. As b is increased and the underlying fractal lattice approaches the Euclidean (triangular) lattice, the crossover behavior of the exponent α from a finite to vanishing values is obtained.

The model treated here is a generalized one of that in [1]. The zero-stage SG resistor network is simply a triangle. The first few stages of construction of the SG fractal family for the members with $b=2$ and $b=3$ are shown in Fig. 1. The fractal of stage n is obtained by aggregating $b(b+1)/2$ cop-

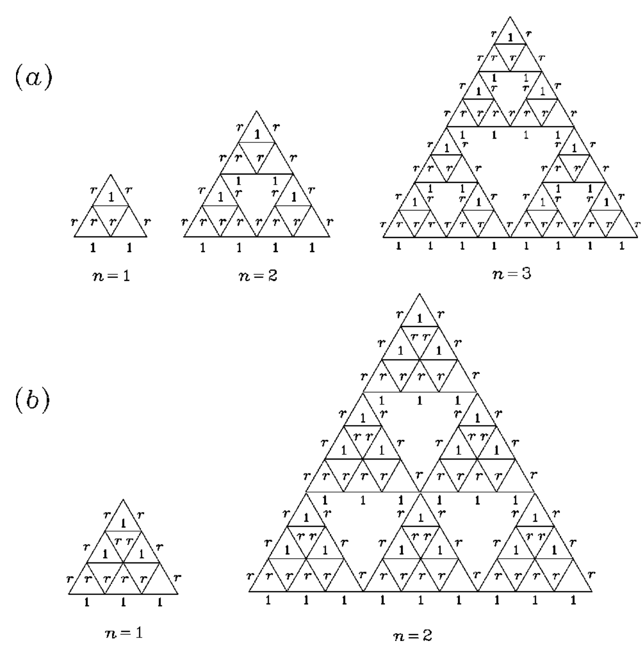

FIG. 1. First two members of the two-dimensional SG type of fractal resistor networks constructed from generators of side length (a) $b=2$, (b) $b=3$. A resistor of resistance 1 is associated with each bond in the horizontal direction while a resistor of resistance r is associated to each of the remaining bonds of the network.

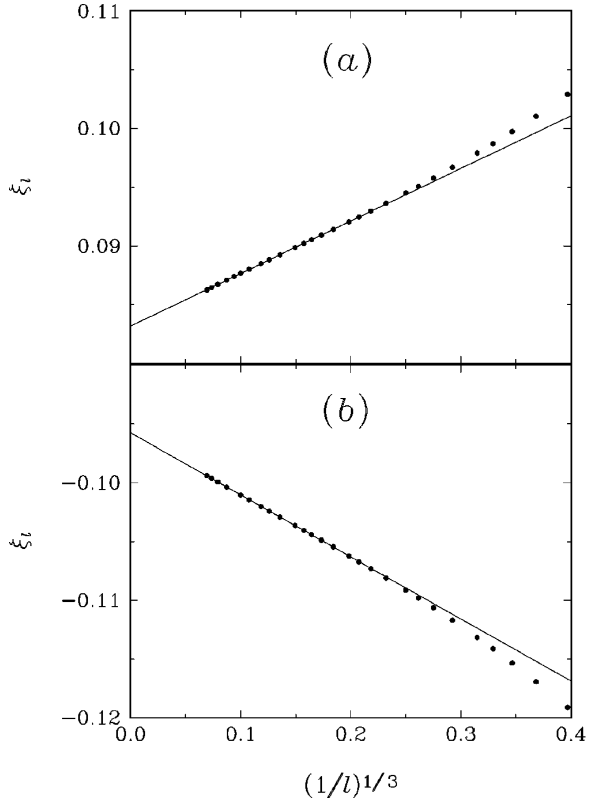

FIG. 2. Finite size scaling behavior of the anisotropy parameter ξ_l for a triangular resistor network with two typical values of basic microscopic anisotropy (a) $r=1.2$ and (b) $r=0.8$. It is seen that the system remains anisotropic on the macroscopic scale.

ies of the $(n-1)$th stage fractals in a way as shown in Fig. 1. For each stage of fractal network, a resistor of resistance 1 is associated with each bond in the horizontal direction while a resistor of resistance r is associated to each of the remaining bonds of the network (see Fig. 1). Here $r\neq 1$ parametrized the degree of basic microscopic anisotropy. Note that if $b=2$, the model reduces to that studied in [1].

By repeated use of the star-triangle (Y-∇) transformation relations [4], any nth-stage fractal network can be reduced to a simple equivalent triangle network. Let X_l denote the resistance for the bond in horizontal direction of an equivalent triangle network that is obtained from an nth-stage fractal network of size $l=b^n$, and Y_l denote the resistance for each of the remaining two bonds in such a triangle. It follows from the definition that, for a zero-stage network,

$$X_1=1, \quad Y_1=r. \tag{1}$$

Define the anisotropy parameter as $\xi_l\equiv\ln(Y_l/X_l)$, which measures the degree of anisotropy of the fractal network that is of size l and composed of a resistance element with basic microscopic anisotropy $Y_1/X_1=r$. Starting from the initial condition (1), the anisotropy parameter ξ_l can be evaluated numerically based on a series of star-triangle transformations. The numerical calculations are performed as follows.

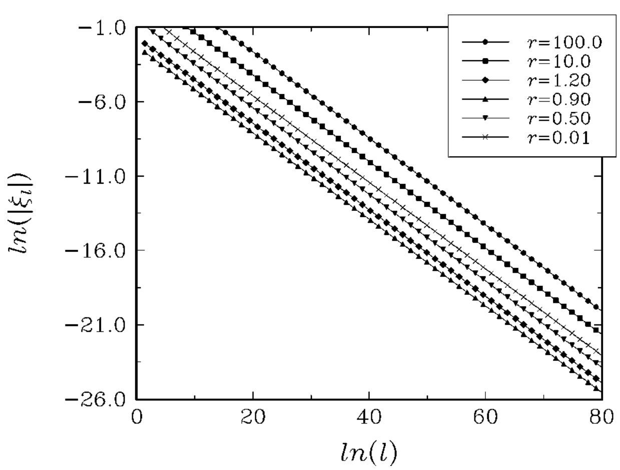

FIG. 3. Plot of $\ln|\xi_l|$ as a function of $\ln l$ for a SG type of fractal resistor network with $b=4$ and different values of r, suggesting that ξ_l satisfies the scaling law $\xi_l\sim l^{-\alpha}$, with the exponent α independent of the value of r.

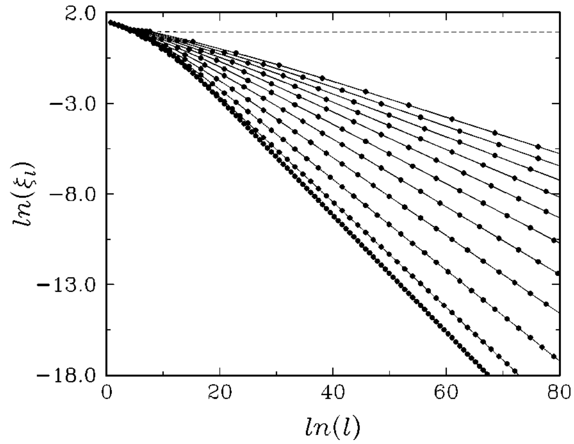

FIG. 4. Plot of $\ln \xi_l$ as a function of $\ln l$ for a SG type of fractal resistor network with $r=100$ and $b=2$, 4, 8, 16, 32, 64, 128, 256, 512, 1024, and 2048 (from lower to upper), showing the underlying structure dependence of the exponent α. The horizontal dashed line is the result for a triangular network inferred from the finite size scaling as shown in Fig. 2.

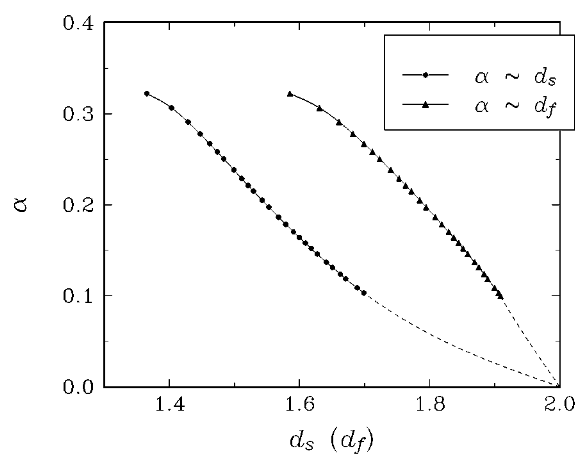

FIG. 5. Plot of the isotropy restoration exponent α as a function of fractal dimension d_f and spectral dimension d_s, showing the fractal-to-Euclidean crossover behavior of the isotropy restoration feature in fractal space. The dashed lines are obtained by numerical interpolation from the calculated results for fractal networks [up to $b=2048$ ($d_f=1.9092$) and $b=1600$ ($d_s=1.6992$) for d_f and d_s dependence of α, respectively] and that for the triangular network, where $d_s=d_f=2$ and $\alpha=0$.

First, we calculate the anisotropy parameter ξ_l as a function of size l for a regular triangular network, which is simply a first stage SG type network with $b=l$. The results are shown in Fig. 2 for two typical values of r. The finite size scaling suggests that the triangular network retain finite anisotropy on the macroscopic scale due to the basic microscopic anisotropy, as is widely believed [1].

Then we evaluate ξ_l for a SG type fractal network with a variety of b and r. Figure 3 shows, as an example, the results for $b=4$ with different values of r. It is observed that the anisotropy parameter ξ_l satisfies the scaling law

$$\xi_l \sim l^{-\alpha}, \tag{2}$$

with the exponent α independent of r. In Fig. 4 we present the results for SG type networks with different values of b, which indicates that the exponent α depends on the underlying fractal structure.

Finally, we study the fractal-to-Euclidean crossover behavior of the isotropy restoration feature by changing the underlying fractal lattice towards the Euclidean triangular lattice. This is achieved by increasing the value of b [1]. The results are shown in Fig. 5 for the exponent α as a function of the fractal dimension d_f as well as the spectral dimension d_s. We have calculated d_f dependence of α up to $b=2048$ ($d_f=1.9092$) and d_s dependence of α up to $b=1600$ ($d_s=1.6992$). Here $d_f=\ln b(b+1)/2/\ln b$, while the spectral dimension d_s is obtained by an *exact* real space renormalization group approach based on the dynamic scaling theory [5,6]. From Fig. 5 the crossover of the isotropy restoration exponent α from fractal space (finite value) to Euclidean space (vanishing value) is observed.

Although the results presented here are specific to a resistor network on finitely ramified fractals, we tend to believe that, with an appropriate definition of anisotropy parameter ξ_l, the scaling law $\xi_l \sim l^{-\alpha}$ can be observed in various physical setups such as diffusion and random walk on a wide class of fractals. As a result, an exponent that characterizes the universal phenomenon of the macroscopic isotropy restoration in fractal space is needed to emphasize the fractal characteristics. In addition, it is also interesting to establish a possible relationship between the isotropy restoration exponent α and the fractal (spectral) dimension as well as other exponents [7] in fractal.

In summary, we have studied the isotropy restoration phenomenon on a family of SG type fractal resistor networks. A parameter is introduced to describe this phenomenon which satisfies the scaling relation $\xi_l \sim l^{-\alpha}$. By changing the underlying fractal lattice towards the Euclidean triangular lattice, the fractal-to-Euclidean crossover of the isotropy restoration exponent α is observed. It is conjectured that this exponent is universal on a wide class of fractals in the sense that it is independent of the degree of microscopic anisotropy. Therefore one more exponent, the isotropy restoration exponent, may be necessary to emphasize the universal phenomenon of macroscopic isotropy restoration in fractals.

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